I. BREVIK

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Synopsis

The covariant conservation laws for a non-closed mechanical system are investigated by emphasizing the interpretation of the work and heat supply as seen from the particular reference frame in which the system is at rest. A distinction is made between the cases where the heat is produced within the system and where it is transferred through the boundary by a convective process. Relativistic transformation formulas for temperature and heat are obtained in general, and the alternative proposals recently put forward in literature are found as corresponding to two special cases. In an appendix, comparisons are made with a result obtained recently by Professor C. Møller.

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I. Introduction

Recently, there has been a considerable discussion on the necessity of revising the formerly accepted formulas

$$\Delta Q = \gamma^{-1}\Delta Q^0,\ T = \gamma^{-1}T^0,\ \left(\gamma = \left(1 - v^2/c^2\right)^{-\frac{1}{2}}\right) \tag{1}$$

in relativistic thermodynamics. Instead of (1), Arzeliès[1], Gamba[2], Kibble[3] and Ott[4] want the following relations

$$\Delta Q = \gamma\Delta Q^0,\ T = \gamma T^0. \tag{2}$$

In the case of an ideal fluid, eq. (1) may be derived[5] by means of the thermodynamical laws

$$\Delta E = \Delta Q + \Delta A\,,\ \Delta S \geq \Delta Q/T \tag{3}$$

and by means of the following assumption for the mechanical work ΔA exerted on the system (for small displacements)

$$\Delta A = -p\Delta V + \boldsymbol{v}\cdot\Delta\boldsymbol{G}\,, \tag{4}$$

where p is the pressure and **G** the total momentum.

From the following investigation it results that the transformation formula for heat is not uniquely determined by the first thermodynamical law but is dependent also on the description of heat supply in the rest inertial system K^0. Moreover, the transformation formula for temperature is found to be given by (2).

2. General Theory

Consider an elastic body which in K^0 is subjected to some *external* forces with resulting density $\boldsymbol{f}^0(\boldsymbol{r}^0, t^0)$ so that each volume element undergoes a

(1) H. Arzeliès, Nuovo Cimento 35, 792 (1965).
(2) A. Gamba, Nuovo Cimento 37, 1792 (1965).
(3) T. W. B. Kibble, Nuovo Cimento 41 B, 72 (1966).
(4) H. Ott, Zeitschr. für Physik 175, 70 (1963).
(5) C. Møller, The Theory of Relativity (Oxford 1952).

certain displacement with a small velocity $\boldsymbol{u}^0(\boldsymbol{r}^0, t^0)$ within the time period $t^0 = O$ to $t^0 = \tau^0$. The displacement itself is not restricted to be small. When $t^0 = \tau^0$, we suppose all elements to be at rest, so that the momentum change is zero. Now, suppose that the body during the time τ^0 is supplied also with an amount of energy $\varDelta Q^0$ of the disordered nature we regard as heat. The components $T^0_{\mu\nu}$ of the mechanical energy-momentum tensor must describe also the effects arising from the heat. The manner in which the heat arises in the body, however, is a crucial point. An amount $\varDelta Q^0_{\text{conv}}$ may flow *through the boundary* by a process of conduction or radiation. We suppose that this amount originates from sources of different (not necessarily only infinitesimally different) temperatures. $\varDelta Q^0_{\text{conv}}$ is counted positive when heat flows into the body. This kind of transfer is described by the energy current components $T^0_{4k} (k = 1, 2, 3)$, while the accompanying momentum density is determined from $T^0_{k4} = T^0_{4k}$ and the momentum flow from T^0_{ik}. We demand explicitly no resulting momentum to be transferred to the body by this flow.

Besides, we shall take into account a heat amount $\varDelta Q^0_{\text{prod}}$ which is assumed to be *produced* within the body, in which case the rate of change $q^0(\boldsymbol{r}^0, t^0)$ of heat density is described by the fourth component of the four-force extended to read $f^0_4 = i/c(\boldsymbol{f}^0 \cdot u^0 + q^0)$. We have then $\varDelta Q^0_{\text{prod}} = \int q^0 dV^0 dt^0$, while $Q^0_{\text{prod}} = \int q^0 dV^0$ means the rate of produced heat in the total body. (Q^0_{conv} has the corresponding meaning for the convective heat.) We assume the heat production to be a dissipative effect, caused for instance by electromagnetic fields in a conductor. Actually this is a kind of work exerted on the body, but we find it natural to include it into $\varDelta Q^0$ rather than into $\varDelta A^0$ in the cases where the amount of energy brought about is clearly connected with a change of entropy.

In order to elucidate this concept of produced heat, let us consider the specific situation where a conductor carries electric current. If $S_{\mu\nu}$ denotes the electromagnetic energy-momentum tensor and $f^{(e)}_\mu$ the electromagnetic force density, then we find by integrating the conservation equation $\partial^0_\nu S^0_{4\nu} = -f^{(e)0}_4$ over the body that the rate of increase of electromagnetic plus heat energy is furnished by a flow of electromagnetic energy through the boundary. However, this energy flow is *not* to be included into T^0_{4k} on the boundary, since it is not a heat radiation between bodies of different temperatures. Instead, as we have remarked, the heat effect is described as originating from f^0_4, which again of course may cause changes in the components of $T^0_{\mu\nu}$ as well as in $\boldsymbol{f}^0$. Since f^0_μ is the resulting external force density, we can write $f^0_\mu = f^{(e)0}_\mu + f^{(m)0}_\mu$, where $f^{(m)0}_\mu$ is the mechanical force of nonelectromagnetic origin. $\boldsymbol{f}^{(m)0}$ will in general include also the reaction force which is necessary in K^0 in order to compensate the electromagnetic force. If the body is kept at rest by external surface forces, then $\boldsymbol{f}^{(m)0} = O$ in the interior domain, but the compensation is achieved by means of the stress tensor T^0_{ik} (this point has been elaborated by OTT[(4)]).

The described production of heat is in principle an irreversible process. Note that the above considerations are valid only when ΔQ^0_{prod} is produced by some *external* sources; if this kind of energy is brought about, e.g. by some chemical change in the body itself, our description is inappropriate.

When $t^0 < O$ and $t^0 > \tau^0$, stationary state is required, i.e. $Q^0 = Q^0_{\text{conv}} + Q^0_{\text{prod}} = O$. Either do then the parts Q^0_{conv} and Q^0_{prod} vanish also, or, as may be the case for a body carrying stationary current, heat is flowing out of the system at the same rate as it is produced, as expressed by $Q^0_{\text{conv}} = -Q^0_{\text{prod}}$.

According to the picture we have now sketched in K^0, the differential conservation laws may be written $\partial^0_\nu T^0_{\mu\nu} = f^0_\mu = (\boldsymbol{f}^0, i/c(\boldsymbol{f}^0\cdot\boldsymbol{u}^0 + q^0))$, and the relativistic transformation formulas we obtain for temperature and heat are the consequences of this picture.

Let us investigate the process as seen from an inertial frame K in which K^0 is moving with the velocity v along the x axis. We denote by Σ_1 the part of the world tube of the body bounded by the spacelike surfaces $t^0 = O$ and $t^0 = \tau^0$, and have then the work ΔA_1 exerted in this domain in K as

$$\Delta A_1 = \int_{\Sigma_1} \boldsymbol{f}\cdot\boldsymbol{u}\,\delta V\delta t\,, \tag{5}$$

where $\boldsymbol{u}$ is the velocity of a moving volume element δV. By transforming each factor in the integrand, we obtain

$$\Delta A_1 = \gamma\int \boldsymbol{f}^0\cdot\boldsymbol{u}^0\delta V^0\delta t^0 + \gamma v\int f^0_x\delta V^0\delta t^0 + \gamma\beta^2\int q^0\delta V^0\delta t^0\,. \tag{6}$$

In the derivation of this formula, we have ignored terms of the second order in u^0/c and have further ignored a term involving the product $u^0_x q^0$. If we had permitted u^0 to be arbitrarily large, then the two first terms in (6) would have remained unchanged, but the last term would have been more complicated. Since $\boldsymbol{f}^0$ is assumed to carry no momentum to the body, we obtain

$$\Delta A_1 = \gamma\int \boldsymbol{f}^0\cdot\boldsymbol{u}^0\delta V^0\delta t^0 + \gamma\beta^2\Delta Q^0_{1\,\text{prod}}\,, \tag{7}$$

where the last term contains the produced heat in Σ_1. Since we are to use the macroscopic laws (3) in K, the system should be compared at two instants corresponding to the hyperplanes $t = t_1$ and $t = t_2$, so that Σ_1 is located in between them. The work $\varDelta A_2$ exerted in K in the domain Σ_2 between $t = t_1$ and $t^0 = O$ is equal to

$$\left.\begin{aligned} \varDelta A_2 = \gamma v \int\limits_{\Sigma_2} \left(f_x^0 + \frac{\beta}{c} q^0 \right) \delta V^0 \delta t^0 = \gamma \beta^2 \int\limits_{V_0^0} x^0 f_x^0 \delta V^0 - \\ - \beta^2 t_1 Q_{2\,\mathrm{prod}}^0 + \frac{\gamma \beta^3}{c} \int\limits_{V_0^0} x^0 q^0 \delta V^0 . \end{aligned}\right\} \tag{8}$$

Here we have integrated over time and have taken the integrals over the surface $t^0 = O$ corresponding to the original volume V_0^0 in K^0. The quantity $Q_{2\,\mathrm{prod}}^0$ means the rate of produced heat in the total body before the deformation period. We assume $q^0 = q^0(\boldsymbol{r}^0)$ but $\partial q^0 / \partial t^0 = O$ outside Σ_1.

A similar expression is obtained for $\varDelta A_3$ corresponding to Σ_3 between $t^0 = \tau^0$ and $t = t_2$, and so the total work $\varDelta A = \varDelta A_1 + \varDelta A_2 + \varDelta A_3$ takes the form

$$\left.\begin{aligned} \varDelta A = \gamma \int \boldsymbol{f}^0 \cdot \boldsymbol{u}^0 \delta V^0 \delta t^0 + \beta^2 [\gamma \varDelta Q_{1\,\mathrm{prod}}^0 - t_1 Q_{2\,\mathrm{prod}}^0 + \\ + (t_2 - \gamma \tau^0) Q_{3\,\mathrm{prod}}^0] + \gamma \beta^2 \left[\int\limits_{V_0^0} - \int\limits_{V^0} \right] x^0 \left(f_x^0 + \frac{\beta}{c} q^0 \right) \delta V^0 . \end{aligned}\right\} \tag{9}$$

Here V^0 is the volume in K^0 for $t^0 \geq \tau^0$. Note that q^0 may have different values within the two integrals in the last term.

The total energy after the deformation may readily be obtained by a tensor transformation as

$$E = 2 \gamma v \int\limits_{V^0} g_x^0 \delta V^0 + \gamma \beta^2 \int\limits_{V^0} T_{xx}^0 \delta V^0 + \gamma E^0 . \tag{10}$$

The first term vanishes, since no total momentum is supplied; moreover, since the energy in K^0 after the deformation is given by

$$E^0 = E_0^0 + \int \boldsymbol{f}^0 \cdot \boldsymbol{u}^0 \delta V^0 \delta t^0 + \varDelta Q^0 , \tag{11}$$

where E_0^0 is the energy before the deformation, we have

$$\varDelta E = \gamma \int \boldsymbol{f}^0 \cdot \boldsymbol{u}^0 \delta V^0 \delta t^0 + \gamma \beta^2 \left[\int\limits_{V^0} - \int\limits_{V_0^0} \right] T_{xx}^0 \delta V^0 + \gamma \varDelta Q^0 . \tag{12}$$

From (12) and (9) we can now find $\varDelta Q$ by the first law. Taking into account that $\int (T_{xx}^0 + x^0 f_x^0) \delta V^0$ is equal to the surface integral $\int x^0 T_{xk}^0 n_k^0 \delta S^0$, and that $\int x^0 q^0 \delta V^0$ may be substituted by $\int x^0 S_k^0 n_k^0 \delta S^0$, we obtain finally the rather unwieldy expression

$$\left.\begin{aligned} \Delta Q = {} & \gamma \Delta Q^0 - \beta^2 [\gamma \Delta Q^0_{1\,\mathrm{prod}} - t_1 Q^0_{2\,\mathrm{prod}} + (t_2 - \gamma \tau^0) Q^0_{3\,\mathrm{prod}}] + \\ & + \gamma \beta^2 \left[\int_{S^0} - \int_{S_0^0} \right] x^0 \left(T^0_{xk} + \frac{\beta}{c} S^0_k \right) n^0_k \delta S^0 . \end{aligned}\right\} \tag{13}$$

Here S^0, S^0_0 are the surfaces corresponding to V^0, V^0_0.

In the following we shall confine ourselves to two cases. In the first place, we consider the situation where no deformation occurs, but where heat is being developed in the body at the same constant rate as it is transferred outwards through the boundary. With $\tau^0 = O$, we obtain from (13), omitting superfluous indices,

$$\Delta Q = -\beta^2 (t_2 - t_1) Q^0_{\mathrm{prod}} = -\gamma \beta^2 \Delta Q^0_{\mathrm{prod}} , \tag{14}$$

where $\Delta Q^0_{\mathrm{prod}}$ is the heat produced in K^0 during a time $\gamma^{-1}(t_2 - t_1)$. This corresponds to replacing the total integration domain Σ by a domain Σ' bounded by two time-orthogonal surfaces in K^0 lying at a distance $\gamma^{-1}(t_2 - t_1)$ apart from each other. The amounts $\Delta Q^0_{\mathrm{prod}}$ corresponding to Σ and Σ' are equal, since q^0 is time-independent at any place.

From (14), (12) and (9) it follows that $\Delta Q < O$, $\Delta E = O$, $\Delta A > O$. Physically this means that the extracted heat must compensate the mechanical work in order to maintain constant energy.

In the second place we assume, as the more interesting case, that both conducted and produced heat is absent in K^0 except in the period $O \leq t^0 \leq \tau^0$. Again omitting a superfluous index, we then obtain from (13)

$$\Delta Q = \gamma \Delta Q^0 - \gamma \beta^2 \Delta Q^0_{\mathrm{prod}} = \gamma \Delta Q^0_{\mathrm{conv}} + \gamma^{-1} \Delta Q^0_{\mathrm{prod}} . \tag{15}$$

We then want to consider the supply of heat as reversible in order to find the transformation formula for temperature from the invariance property of entropy. Since $\Delta Q^0_{\mathrm{prod}}$ arises on account of an irreversible process, we have to put this quantity equal to zero and demand $\Delta Q^0_{\mathrm{conv}}$ to be transferred reversibly. Assuming the temperature to change negligibly during the process in any reference system, we then obtain the formula

$$T = \gamma T^0 . \tag{16}$$

Next, assume the situation corresponding to (15). We want to compare formula (9) in this case with formula (4), commonly accepted as valid for an ideal fluid. The term $-p\Delta V = -p^0\gamma^{-1}\Delta V^0$ corresponds to $\gamma^{-1}\int \boldsymbol{f}^0\cdot\boldsymbol{u}^0\delta V^0\delta t^0$; moreover, we obtain by the same method as led to (12)

$$\Delta G_x = \frac{\gamma\beta}{c}\left[\int \boldsymbol{f}^0\cdot\boldsymbol{u}^0\delta V^0\delta t^0 + \left(\int\limits_{V^0} - \int\limits_{V_0^0}\right)T_{xx}^0\delta V^0 + \Delta Q^0\right], \tag{17}$$

so that for small displacements

$$\Delta A = -p\Delta V + \boldsymbol{v}\cdot\Delta\boldsymbol{G} - \gamma\beta^2\Delta Q_{\mathrm{conv}}^0 \tag{18}$$

where ΔA is determined by (9). Eq. (4) is thus found to be valid only in certain cases.

3. Discussion and applications

In some textbooks, when evaluating the work exerted on a fluid in motion, the last term in (4) is claimed to be present since a force $d\boldsymbol{G}/dt$ should be necessary to maintain constant velocity. However, the total force may be put equal to $d\boldsymbol{G}/dt$ only when *no momentum* is transferred through the boundary. This is correct only when $\Delta Q_{\mathrm{conv}}^0 = O$, in which case we have already seen that (18) is equivalent to (4). When all heat thus is produced in the body, formula (1) for ΔQ is valid. But the corresponding formula for T is of course not valid.

Horizontal rod.

A uniform rod of length $2L^0$ and width unity is placed parallel to the x axis and is in K^0 subjected to a uniform pressure p^0 in its length direction so that the work done in the period $t^0 = O$ to $t^0 = \tau^0$ is $\Delta A^0 = -p^0\Delta V^0 = -2p^0\Delta L^0$. This situation, as it appears in K, has been the subject of some discussion. Kibble (3) calculates the work exerted by the applied forces and finds $\Delta A = -p\Delta V$, which coincides with (9) if one ignores the terms associated with the produced heat. Therefore Kibble attains the result (2), as expected. Arzeliès (1) claims that the work exerted in K must be equal to $\gamma\Delta A^0$, that is the quantity which we formerly called ΔA_1 (except from the produced heat term). However, as pointed out by Kibble, by investigating the macroscopic thermodynamical relations, we should work with quantities referring to surfaces of simultaneity in K rather than in K^0. R. Penney (6) adopts Arzeliès' point of view, but claims additional forces to be present in

(6) R. Penney, Nuovo Cimento 43, 911 (1966).

the transient periods in order to maintain constant speed. This way of reasoning, however, means in general that one leaves the expression for the work density per unit time as $\boldsymbol{f}\cdot\boldsymbol{u}$, where $\boldsymbol{f}$ is the real force, obtained by a relativistic transformation from K^0. We shall mention also another paper by Arzeliès(7), in which he considers a material particle and claims that a bare supply of heat in $K^0(\varDelta A^0 = 0)$ must be equivalent to a bare supply of heat in K. This point of view is implicitly the same as that of Kibble, and leads to the same result as his. We agree that this should be the more normal situation in thermodynamics.

Vertical rod.

When no heat is taken into account, we find in K $\varDelta A = \gamma\varDelta A^0 = -\gamma^2 p\varDelta V$. Of course this should not be compared with the formula $\varDelta A = \gamma^{-1}\varDelta A^0 + \gamma\beta^2\varDelta(E^0 + p^0 V^0)$, which for vanishing heat is valid for a fluid with *isotropic* pressure and incidently is valid also for the horizontal rod.

Penney(6) wants to exclude the work performed by the horizontal force F_x in K because of its lack of physical reality. This effect is claimed to be balanced by the internal pressure. It therefore seems as desirable to write down the information we can obtain from the energy conservation equation in K in integrated form

$$\int \boldsymbol{\sigma}\cdot\boldsymbol{n}\delta S + \frac{d}{dt}\int h\delta V = \int \boldsymbol{f}\cdot\boldsymbol{u}\delta V \tag{19}$$

Here the integration volume follows the body and contains the part which lies between $y = O$ and an arbitrary $y < L$. (In K^0 the rod is placed in such a way that $O \leq y \leq 2L$). h is the energy density and $\boldsymbol{\sigma} = S - h\boldsymbol{u}$ the relative energy current. Let the velocity of the lower end in K^0 be denoted by u_0^0; by ignoring second order quantities we then obtain

$$\int \boldsymbol{\sigma}\cdot\boldsymbol{n}\delta S = p^0 u_0^0(1 - y/L), \; \frac{d}{dt}\int h\delta V = p^0 u_0^0 y/L, \tag{20}$$

and so the total work $p^0 u_0^0$ per unit time is seen partly to yield an increase of energy, partly to cause an energy flow through the boundary.

Remarks on Ott's work(4)

This long and comprehensive paper deserves special attention, so let us dwell on two essential points.

(7) H. Arzeliès, Nuovo Cimento 40B, 333 (1965).

1. Consider a material particle which in K^0 receives a convective supply of heat. In our earlier treatment we have let such a kind of transfer be described by the tensor components T^0_{4k}, but Ott lets the change of heat energy be described by an additional term in the fourth component of the four-force. Further, in accordance with the inertia of energy, this heat flow is accompanied by a momentum density whose rate of change in integrated form is represented by extra terms in the ordinary force. Ott obtains the change ΔQ in K by a four-vector transformation from K^0, and finds that $\Delta Q = \gamma \Delta Q^0$ in general, also when an external force $\boldsymbol{F}^0$ present in K^0 supplies the body with the mechanical momentum $\int \boldsymbol{F}^0 dt^0$ during the actual period τ^0. The situation considered earlier in our paper corresponds to $\int \boldsymbol{F}^0 dt^0 = O$, and with this restriction we see that Ott's result agrees with (15) since the last term in (15) is absent. (See also the end of the appendix).

In his applications, Ott considers extended systems also and obtains the same formula (2) for ΔQ. It is assumed that a transfer of heat into the system is not associated with the occurrence of a mechanical force in a reference frame in which the system is moving. It may be seen readily that with this assumption, Ott's results agree with those we obtained earlier in section 2. For the force density is restricted so as to contain no heat term, and thus produced heat is absent and the formula (15) takes the simplified form (2).

2. As the second point, let us again consider the stationary situation in which a constant electric current is flowing through a conductor and develops heat in K^0 at the same rate as it is transferred outwards through the boundary, i.e. $Q^0_{\text{prod}} = -Q^0_{\text{conv}}$, q^0 being independent of time at any place. In another reference frame, Ott obtains $q = \gamma q^0$, which seems incompatible with our earlier results for the transformation of produced heat. Let us therefore make a closer investigation on this situation.

The first law may be obtained in general by means of the integrated form of the differential conservation equation involving the fourth line of the mechanical energy-momentum tensor. By adding the produced heat-term to (19) and integrating over time, we obtain

$$\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \delta S \delta t + \Delta \int h \delta V = -ic \int f_4 \delta V \delta t = \int \boldsymbol{f} \cdot \boldsymbol{v} \delta V \delta t + \int q \delta V \delta t, \tag{21}$$

or

$$-\Delta Q_{\text{conv}} + \Delta E = \Delta A + \Delta Q_{\text{prod}}. \tag{22}$$

In the present case $\Delta E = O$. The essential point here is that we have *assumed the form* $f_4 = i/c(\boldsymbol{f}\cdot\boldsymbol{v}+q)$ in any frame, where $\boldsymbol{f}\cdot\boldsymbol{v}$ is interpreted as the mechanical work exerted per unit of volume and time. By this assumption, we obtain at once by Lorentz-transformations from K^0, $q^0 = -icf_4 - \boldsymbol{f}\cdot\boldsymbol{v} = \gamma^{-1}q^0$. If we integrate over a domain Σ in four-space which lies between two constant time-surfaces in K, we obtain $\Delta Q_{\text{prod}} = \gamma^{-1}\Delta Q^0_{\text{prod}}$, where at the right hand side we again have substituted Σ by Σ' as explained in the section following (14). By means of (14) we obtain $\Delta Q_{\text{conv}} = \Delta Q - \Delta Q_{\text{prod}} = -\gamma\Delta Q^0_{\text{prod}} = \gamma\Delta Q^0_{\text{conv}}$. Note that this transformation formula for the convective heat has the same form as (2).

Let us investigate the possibility of splitting f_4 into two terms in other ways than shown above, corresponding to different interpretations of the mechanical work. An actual possibility might be to put $f_4 = i/c(\boldsymbol{f}'\cdot\boldsymbol{v}+q')$, where the four-vector f'_μ is determined by $f'^0_\mu = (\boldsymbol{f}^0, O)$. Thus the mechanical work density per unit time is interpreted as $\boldsymbol{f}'\cdot\boldsymbol{v}$, which means that we split off the effect arising from the heat term in f^0_4. By means of the covariant relation $f_\mu = f'_\mu + q^0 V_\mu/c^2$, where $V_\mu = \gamma(\boldsymbol{v}, ic)$ is the four-velocity of the medium, we obtain readily $q' = \boldsymbol{f}\cdot\boldsymbol{v} - \boldsymbol{f}'\cdot\boldsymbol{v} + q = \gamma q^0$, which agrees with Ott's result. The first law can still be written as (21), when $\boldsymbol{f}$ and q are replaced by $\boldsymbol{f}'$ and q'.

However, the main obstacle for the present interpretation is apparent from the conservation equations $\partial_\nu T_{i\nu} = f'_i + q^0 V_i/c^2$. A momentum component inside a certain volume element may change on account of a momentum flow (described by T_{ik}) and a mechanical force (described by f'_i). But there is a lack of interpretation of the last term to the right. The common splitting of f_4—the one used throughout this paper—seems to be preferable from a physical point of view.

Actually, Ott proceeds in another way in order to find the developed heat density. He claims the rate of mechanical work density to be equal to $\boldsymbol{k}\cdot\boldsymbol{v}$, where the four-vector k_μ is determined by $k^0_\mu = (\boldsymbol{k}^0, 0)$, $k^0_i = -\partial^0_k T^0_{ik}$, i.e. $\boldsymbol{k}^0$ is the elastic force density which in the interior domain of the body just compensates the electromagnetic force, so that $\boldsymbol{k}^0 + \boldsymbol{f}^{(e)0} = O$. This performance does not seem to be readily incorporated in the above formalism, however, as we have let f_μ denote *external* forces.

Finally, we mention that in order to obtain the transformation formula for q we may consider instead the energy balance equation for the electromagnetic field, $\partial_\nu S_{4\nu} = -f^{(e)}_4 = -i/c(\boldsymbol{f}^{(e)}\cdot\boldsymbol{v}+q)$, where $S_{\mu\nu}$ is the electromagnetic energy momentum tensor. We obtain of course $q = \gamma^{-1}q^0$ as before. By adding the equations belonging to the two systems, however, we find $\partial_\nu(T_{4\nu}+S_{4\nu}) = f_4^{(m)}$, where $f_\mu^{(m)0} = (\boldsymbol{f}^{(m)0}, 0)$ so that the heat terms eliminate each other, and we obtain from this equation neither the first law nor the transformation formula for heat.

I wish to thank Professor C. Møller for valuable discussions. The reader is referred also to a paper by Professor Møller[8] on the present subject. That paper was written during his absence from Copenhagen in the autumn of 1966.

(8) C. Møller, Relativistic Thermodynamics. A Strange Incident in the History of Physics Mat. Fys. Medd. Dan Vid. Selsk. 36, no. **1** (1967).

Nordita, Copenhagen.

Appendix

Let us write down the integrated conservation equations for momentum

$$\int t_{ik} n_k \delta S \delta t + \left(\int_{V^0} - \int_{V_0^0} \right) g_i \delta V = \int f_i \delta V \delta t \,, \tag{A. 1}$$

where $t_{ik} = T_{ik} - g_i u_k$. Eq. (A. 1) can also be written as

$$\Delta G_i = \Delta G_i^{(m)} + \Delta G_i^{(h)}, \tag{A. 2}$$

where

$$\Delta G_i^{(m)} = \int f_i \delta V \delta t \tag{A. 3}$$

denotes the increase of momentum due to the external forces and

$$\Delta G_i^{(h)} = - \int t_{ik} n_k \delta S \delta t \tag{A. 4}$$

the increase of momentum due to the heat flow through the boundary. It has been shown by C. Møller[8] that, in the special example considered in his paper, the quantities

$$\Delta Q_\mu = \left(\Delta \boldsymbol{G}^{(h)}, \frac{i}{c} \Delta Q \right) \tag{A. 5}$$

form a four-vector. It therefore seems to be an interesting point to make use of the above formalism to investigate whether the four-vector property of ΔQ_μ is more generally valid.

Let us write down the assumptions. In the first place, any produced heat is required to be absent. This means that no heat is supplied except when $0 \leq t^0 \leq \tau^0$. In the second place, the velocity u^0 of a volume element in K^0 is permitted to be arbitrarily large. This is a weaker restriction than we had before in our paper, and means that more general irreversible processes will be included in the description.

Let us now find $\Delta G_i^{(h)}$. The component ΔG_x is still given by (17), and we readily find that

$$\Delta G_y = \frac{\beta}{c}\left(\int\limits_{V^0} - \int\limits_{V_0^0}\right) T_{yx}^0 \delta V^0, \Delta G_z = \frac{\beta}{c}\left(\int\limits_{V^0} - \int\limits_{V_0^0}\right) T_{zx}^0 \delta V^0 . \qquad (A.\ 6)$$

Further

$$\Delta G_x^{(m)} = \int\limits_{\Sigma} f_x \delta V \delta t = \gamma \int\limits_{\Sigma} f_x^0 \delta V^0 \delta t^0 + \frac{\gamma\beta}{c} \int\limits_{\Sigma_1} \boldsymbol{f}^0 \cdot \boldsymbol{u}^0 \delta V^0 \delta t^0 . \qquad (A.\ 7)$$

The first integral on the right hand side is to be evaluated over $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$. The contribution from Σ_1 is $\Delta G_x^{(m)0}$, while the contributions from Σ_2 and Σ_3 may be found by performing the time integrations for each volume element. We obtain

$$\Delta G_x^{(m)} = \frac{\gamma\beta}{c}\left[\Delta A^0 + \left(\int\limits_{V_0^0} - \int\limits_{V^0}\right) x^0 f_x^0 \delta V^0\right] + \gamma \Delta G_x^{(m)0}, \qquad (A.\ 8)$$

where $\Delta G_x^{(m)0} = -\Delta G_x^{(h)0}$.

Similarly,

$$\Delta G_y^{(m)} = \int\limits_{\Sigma} f_y^0 \delta V^0 \delta t^0 = \frac{\beta}{c}\left[\int\limits_{V_0^0} - \int\limits_{V^0}\right] x^0 f_y^0 \delta V^0 + \Delta G_y^{(m)0} \qquad (A.\ 9)$$

$$\Delta G_z^{(m)} = \frac{\beta}{c}\left[\int\limits_{V_0^0} - \int\limits_{V^0}\right] x^0 f_z^0 \delta V^0 + \Delta G_z^{(m)0} . \qquad (A.\ 10)$$

From (17) and (A. 6–10) we obtain, after some partial integrations,

$$\left.\begin{aligned} \Delta G_x^{(h)} &= \gamma \Delta G_x^{(h)0} + \frac{\gamma\beta}{c} \Delta Q^0 \\ \Delta G_y^{(h)} &= \Delta G_y^{(h)0}, \ \Delta G_z^{(h)} = \Delta G_z^{(h)0} \end{aligned}\right\} \qquad (A.\ 11)$$

The work ΔA follows from (6)–(9) as

$$\Delta A = \gamma \Delta A^0 + \gamma\beta^2 \left[\int\limits_{V_0^0} - \int\limits_{V^0}\right] x^0 f_x^0 \delta V^0 + \gamma v \Delta G_x^{(m)0} \qquad (A.\ 12)$$

(note that u^0 has been considered small only in the derivation of the last term in (6)). The energy change ΔE is still given by (12), and so

$$\Delta Q = \Delta E - \Delta A = v\gamma \Delta G_x^{(h)0} + \gamma \Delta Q^0 . \qquad (A.\ 13)$$

From (A. 13,11) we see that ΔQ_μ transforms like a four-vector between K^0 and K, under the assumptions written down above.

Combining (A. 1) and (21) we may write

$$\Delta G_\mu = \int f_\mu \delta V \delta t - \int t_{\mu k} n_k \delta S \delta t = \Delta G_\mu^{(m)} + \Delta Q_\mu , \qquad \text{(A. 14)}$$

where $t_{\mu k} = T_{\mu k} - g_\mu u_k$. This equation is in general not a relation between covariant quantities, since only the last term to the right is a four-vector. (The quantity $\Delta G_\mu^{(m)}$ is of course a four-vector as long as we let the integration domain be unchanged under Lorentz transformations. However, when we let the time-orthogonal surfaces in K be substituted by appropriate time-orthogonal surfaces in K^0 upon the transformation $K \to K^0$, the four-vector property is in general lost). But in the special case where the system is a material particle, then all terms in (A. 14) are four-vectors. Introducing the element of proper time $d\tau = \gamma^{-1} dt$ for the single world line we are considering, we can write

$$\frac{dG_\mu}{d\tau} = F_\mu + \frac{dQ_\mu}{d\tau}, \qquad \text{(A. 15}$$

where F_μ is the external total four-force. Eq. (A. 15) is the equation utilized by Ott[4] in his general proof. However, his result (2) agrees with (A. 13) only when $\Delta G_x^{(h)0} = O$.

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